Solution to Final Exam, STAT 5585 — Mathematical Statistics I, Fall 2008

1. (a) Let S be the common support and consider the continuous case. By definition of expectation, $E[h(Y_i)f(Y_i)/g(Y_i)] = \int_s \frac{h(y)(f(y)}{g(y)} g(y)\mathrm{d}y = \int_s h(y)(f(y)\mathrm{d}y = E[h(X)] = \theta$.
   (b) Use the stronger version of the weak law of large number.
   (c) Note that $\tilde{\theta}_n = \theta_n / A_n$, where $A_n = \sum_{i=1} f(Y_i)/g(Y_i)/n \xrightarrow{p} E[f(Y_i)/g(Y_i)] = 1$.

2. (a) Note that $G(y_1) = \Phi(x_1)$. Differentiation on both sides gives $g(y_1)\mathrm{d}y_1 = \phi(x_1)\mathrm{d}x_1$.
   (b) The determinant of the Jocobian is $\frac{g(y_1)g(y_2)}{\phi(x_1)\phi(x_2)}$. The joint pdf is

$$h(y_1, y_2) = f(x_1, x_2)\frac{g(y_1)g(y_2)}{\phi(x_1)\phi(x_2)}$$

   which simplifies to

$$h(y_1, y_2) = \frac{g(y_1)g(y_2)}{\sqrt{1-\rho^2}} \exp[-\frac{\rho^2 x_1^2 - 2\rho x_1 x_2 + \rho^2 x_2^2}{2(1-\rho^2)}],$$

   where $x_i = \Phi^{-1}\{G(y_i)\}$.
   (c) The marginal pdf of Y_2 is g. Therefore, the conditional pdf is

$$l(y_1|y_2) = \frac{g(y_1)}{\sqrt{1-\rho^2}} \exp[-\frac{\rho^2 x_1^2 - 2\rho x_1 x_2 + \rho^2 x_2^2}{2(1-\rho^2)}],$$

   where $x_i = \Phi^{-1}\{G(y_i)\}$.

3. (a) By CLT and Delta method, $\sigma^2 = e^{-2\lambda}\lambda$.
   (b) Note that Y_i's are iid Bernoulli with success rate $e^{-\lambda}$. By CLT, $\tau^2 = \mathrm{Var}(Y_1) = e^{-\lambda}(1 - e^{-\lambda})$.
   (c) The ratio $\sigma^2/\tau^2 < 1$. This means that the asymptotic variance is smaller in (a) than in (b).

4. (a) Expand $\sum(X_i - \bar{X}_n + \bar{X}_n - \mu)^2$.
   (b) Note that $\sum(X_i - \mu)^2/n \xrightarrow{p} \sigma^2$ by WLLN. Using (a), $\frac{n-1}{n}S_n^2 = \sum(X_i - \mu)^2/n - (\bar{X}_n - \mu)^2$. The second part converges to zero in probability.
   (c) Let $Y_i = (X_i - \mu)^2$. By CLT, $\sqrt{n}(\bar{Y}_n - \sigma^2)/(\mu_4 - \sigma^4) \xrightarrow{d} N(0, 1)$. Note that

$$\frac{\sqrt{n}(S_n^2 - \sigma^2)}{\sqrt{\mu_4 - \sigma^4}} = \frac{\sqrt{n}(\frac{n-1}{n}S_n^2 - \sigma^2) - \sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu_4 - \sigma^4}}.$$

   It sufficies to show that $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{p} 0$, which is a result of Markov inequalty; see Example 5.2.10 in the textbook.

5. (a) Integrate out y in $f(x, y)$.
   (b) Find first $E[XY] = a\mu_1\xi_1 + (1-a)\mu_2\xi_2$. Find next $E[X]$ and $E[Y]$ from the marginal density.
   (c) Independence is equivalent to $f(x, y) = f_X(x)g_Y(y)$. Simplify to get the desired expression.

THE END