Midterm 1 Solution, STAT 5685 — Mathematical Statistics II, Spring 2009

1. (10 points)

   (a) (2 points) The distribution belongs to a one-parameter exponential family.

   (b) (3 points) Differentiate the log density twice and take negative expectation. The Fisher information is $nI_{X_1}(\theta) = n/\theta^2$.

   (c) (2 points) The two points shows that T is not a function of S, implyint that S is not a sufficient statistic.

   (d) (3 points) Since T is complete sufficient, it is minimal sufficient. Since S is not a function of T, it is not sufficient. Therefore, $I_S(\theta) < I_T(\theta)$.

2. (10 points)

   (a) (2 points) By Neymans' factorization theorem.

   (b) (3 points) By Lehmann-Scheffe Theorem.

   (c) (2 points) Note that $X_i - \theta$ are iid exponential with mean θ and $X_{n:1} - \theta$ is exponential with mean $1/\theta$. $E[T] = \big((1 + \frac{1}{n})\theta, 2n\theta\big)$.

   (d) (3 points) Consider $h(t) = (1 + \frac{1}{n})^{-1}t_1 - \frac{1}{2n}t_2$.

3. (10 points)

   (a) (2 points) From Student's theorem, T_1 is $N(\mu_1, \sigma^2/m)$, T_2 is $N(\mu_2, \sigma^2/n)$, and T_3 is $\Gamma(\frac{m+n-2}{2}, 2\sigma^2)$.

   (b) (3 points) From Student's theorem, the three elements of T are mutually independent. The joint distribution is the product of the three marginals.

   (c) (2 points) The joint density can be written as, for some function a and g,

$$f(t_1, t_2, t_3|\mu_1, \mu_2, \sigma^2) = a(t_1, t_2, t_3)g(\mu_1, \mu_2, \sigma^2)\exp\left(-\frac{mt_1^2 + nt_2^2 + t_3}{2\sigma^2} + \frac{mt_1\mu_1}{\sigma^2} + \frac{nt_2\mu_2}{\sigma^2}\right).$$

   (d) (3 points) By Theorem 6.6.2, $S = (mt_1^2 + nt_2^2 + t_3, mt_1, nt_2)$ is complete and sufficient for $\eta = (1/\sigma^2, \mu_1/\sigma^2, \mu_2/\sigma^2)$. Since there is a one-to-one map between η and θ, and a one-to-one-map between S and T, T is complete and sufficient for θ. Then apply Basu's theorem, since S is ancillary.

4. (10 points)

   (a) (2 points) $E[T] = E[I(X_1 > a)] = \Pr(X_1 > a)$.

   (b) (3 points) Bivariate normal

$$N\left(\begin{bmatrix}\mu \\ n\mu\end{bmatrix}, \begin{bmatrix}1 & 1 \\ 1 & n\end{bmatrix}\right).$$

   (c) (2 points) The conditional distribution of X_1 given $U = u$ is $N(\frac{1}{n}u, 1 - \frac{1}{n})$.

   (d) (3 poitns)

$$E[T|U] = \Pr(X_1 > a|U) = 1 - \Phi\left(\frac{a - \frac{1}{n}U}{\sqrt{1 - \frac{1}{n}}}\right),$$

   where Φ is the cdf of $N(0, 1)$.

THE END