Midterm 2 Solution, STAT 5685 — Mathematical Statistics II, Spring 2009

1. (a) (3 points) Solve the score equation and check second derivative of the loglikelihood to find $\hat{\theta} = \bar{X}_n$.

   (b) (3 points) The likelihood ratio is simplified to be
$$\Lambda = \bar{X}_n^{-n\bar{X}_n} \exp(n\bar{X}_n - n).$$
   Note that $g(t) = nt - n - nt \log t$ increases first and then decreases as t increases; check this by $g'(t)$. Therefore, $\Lambda < k$ is equivalent to $\bar{X}_n < a$ or $\bar{X}_n > b$.

   (c) (4 points) By central limit theorem, under H_0, $Z_n = \sqrt{n}(\bar{X}_n - 1)/sqrt1 \xrightarrow{d} N(0,1)$. A level α test rejects H_0 if $|Z_n| > \Phi^{-1}(1-\alpha/2)$, where Φ is the cdf of $N(0,1)$.

2. (a) (3 points) Apply Neyman's factorization theorem.

   (b) (3 points) The distribution of $\theta^{-1}(k\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i)$ is $\Gamma(n+m, k)$, completely known.

   (c) (4 points) From the pivotal quantity, find a and b such that
$$\Pr(a < \theta^{-1}(k\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i) < b) = 1 - \alpha.$$
   Then a $1-\alpha$ confidence interval is $((k\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i)/b, (k\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i)/a)$.

3. (a) (2 points) For $\theta^* > \theta$,
$$\frac{L(\theta^*)}{L(\theta)} = \left(\frac{\theta}{\theta^*}\right)^n \frac{I(0 < X_{n:n} < \theta^*)}{I(0 < X_{n:n} < \theta)}.$$
   This ratio is a constant for $X_{n:n} < \theta$ and infinity for $\theta < X_{n:n} < \theta^*$. Therefore, it is nondecreasing in $X_{n:n}$.

   (b) (2 points) By Karlin-Rubin Theorem, the UMP test rejects H_0 if $X_{n:n} > k$, where k is such that $\Pr_{\theta_0}(X_{n:n} > k) = \alpha$. With the cdf of $X_{n:n}$ under H_0, k is solved to be $\theta_0(1-\alpha)^{1/n}$.

   (c) (3 points) $\Pr_\theta(X_{n:n} > k) = 1 - (\theta_0/\theta)^n(1-\alpha)$.

   (d) (3 points) The acceptance region of the UMP test with level α is given by $\{X_{n:n} < \theta_0(1-\alpha)^{1/n}$. A one-sided $1-\alpha$ confidence interval contains all θ values such that $X_{n:n} < \theta(1-\alpha)^{1/n}$, which is $(X_{n:n}/(1-\alpha)^{1/n}, \infty)$.

4. (a) (2 points) $T = (\bar{X}_n, S^2_{x,n}, \bar{Y}_m, S^2_{y,m})$.

   (b) (2 points) $\bar{X}_n - \bar{Y}_m$ because it is unbiased and a function of T.

   (c) (3 points) Note that $E[\sqrt{n-1}S_{x,n}/\sigma] = \kappa_{n-1,1/2}$ and that $E[\left(\sqrt{m-1}S_{y,m}/\sigma\right)^{-1}] = \kappa_{m-1,-1/2}$. They lead UMVUE of σ_x and $1/\sigma_y$. From the independence of the two samples, the UMVUE of σ_x/σ_y is
$$\frac{\sqrt{n-1}S_{x,n}}{\sqrt{m-1}\kappa_{n-1,1/2}\kappa_{m-1,-1/2}S_{y,m}}.$$

   (d) (3 points) When $\sigma_x^2 = \sigma_y^2$, a complete and sufficient statistic is $U = (\bar{X}_n, \bar{Y}_m, S_p^2)$, where S_p^2 is the pooled sample variance. Note that $E[\left(\frac{\sqrt{n+m-2}S_p}{\sigma_x}\right)^{-1}] = \kappa_{n+m-2,-1/2}$. By independence of $(\bar{X}_n, \bar{Y}_m)$ and S_p^2, which can be established from Student's theorem, the UMVUE of $(\mu_x - \mu_y)/\sigma_x$ is
$$\frac{\bar{X}_n - \bar{Y}_m}{\sqrt{n+m-2}\kappa_{n+m-2,-1/2}S_p}.$$

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