Quiz 1, STAT 316 Mathematical Statistics II, Spring 2008

Name: **Points:**

1. (10 points) Let $X_1, \ldots, X_n$ be an iid sample from $N(\theta, \theta)$, where $\theta > 0$ is unknown.

   (a) (5 points) Does $N(\theta, \theta)$ belong to the exponential family? Justify.

   (b) (5 points) Derive a minimal sufficient statistic for θ.

   Solution:

   (a) Yes.
   $$f(x;\theta) = \exp\left[-\frac{x^2}{2\theta} + x - \frac{\theta}{2} - \log(\sqrt{2\pi\theta})\right].$$

   (b) Since the distribution is a member of exponential family, a minimal sufficient statistic is $\sum X_i^2$.

   $\square$

2. (10 points) Let $X_1, \ldots, X_n$ be an iid sample from $N(\mu, \sigma^2)$, $\mu \in R$, $\sigma^2 > 0$.

   (a) (5 points) Find a joint sufficient statistic for (μ, σ^2).

   (b) (5 points) Find a sufficient statistic for σ^2 when μ is known.

   Solution:

   (a) From results for exponential family, $(\sum X_i, \sum X_i^2)$ is a joint sufficient statistic. Any one-to-one transformation of it, such as $(\bar{X}, S^2)$, is also joint sufficient statistic.

   (b) With μ known, $\sum(X_i - \mu)^2$ is a sufficient statistic.

   $\square$

THE END

Quiz 2, STAT 316 Mathematical Statistics II, Spring 2008

Name: **Points:**

1. (10 points) Let X be a Bernoulli random variable with success rate $\theta \in (0,1)$.
   (a) (5 points) Find $I_X(\theta)$, the Fisher information about θ in X.
   (b) (5 points) Let $\eta = g(\theta)$ be a differentiable one-to-one transformation of θ. Find $I_X(\eta)$, the Fisher information about η in X.

Solution:

(a)

$$\frac{\partial \log f}{\partial \theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}.$$

$$\frac{\partial^2 \log f}{\partial \theta^2} = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}.$$

$$I_X(\theta) = -E\left[\frac{\partial^2 \log f}{\partial \theta^2}\right] = \frac{1}{\theta(1-\theta)}.$$

(b)

$$I_X(\eta) = E\left[\left(\frac{\partial \log f}{\partial \eta}\right)^2\right] = E\left[\left(\frac{\partial \log f}{\partial \theta}\frac{\partial \theta}{\eta}\right)^2\right] = \left(\frac{\partial \theta}{\eta}\right)^2 I_X(\theta).$$

□

2. (10 points) Let $X_1, \ldots, X_n$ be an iid sample from $\Gamma(\alpha, \beta)$, where α is known. Let $T = \sum_{i=1}^n X_i$.
   (a) (5 points) Show that $W = (X_{n:n} - X_{n:1})/T$ is an ancillary statistic.
   (b) (5 points) Find $\mathrm{Cov}(W, T)$.

Solution:

(a) $\Gamma(\alpha, \beta)$ is a scale family with scale parameter β.

(b) $\Gamma(\alpha, \beta)$ is an exponential family. Therefore, T is a complete and sufficient statistic. By Basu's Theorem, T and W are independent and $\mathrm{Cov}(W, T) = 0$.

□

THE END

Solution to Quiz 3, STAT 316 Mathematical Statistics II, Spring 2008

Name: **Points:**

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1. (10 points) Suppose that $X_1, \ldots, X_n$ form a random sample from $U(0, \theta)$. Let $X_{n:n}$ be the largest order statistic.

   (a) (5 points) Find the UMVUE of θ^2.

   (b) (5 points) Find $E[X_1^2 | X_{n:n}]$.

   Solution:

   (a) Since $X_{n:n}$ is complete and sufficient statistic, it suffices to find a function of $X_{n:n}$ whose expectation is θ^2. Start from a guess $X_{n:n}^2$ and find that $E[X_{n:n}^2] = \frac{n}{n+2}\theta^2$, knowing that the density of $X_{n:n}$ is $g(t) = nt^{n-1}/\theta^n$, $0 < t < \theta$.
   Therefore, $\frac{n+2}{n}X_{n:n}^2$ is UMVUE of θ^2 by Lehmann-Scheffe.

   (b) By Lehmann-Scheffe, $E[X_1^2 | X_{n:n}]$ is the UMVUE of $E[X_1^2] = \frac{\theta^2}{3}$, which is, by the above result, $\frac{n+2}{3n}X_{n:n}^2$.

   □

2. (10 points) Suppose that $X_1, \ldots, X_n$ form a random sample from $N(\theta, 1)$.

   (a) (5 points) Find the UMVUE of θ.

   (b) (5 points) Does the variance of the UMVUE attain the Cramér-Rao lower bound?

   Solution:

   (a) Since $\bar{X}_n$ is complete and sufficient, and $E[\bar{X}_n] = \theta$, the UMVUE if $\bar{X}_n$.

   (b) The CRLB is, after some algebra and taking expectation, found to be $1/n$, which is attained by the variance of $\bar{X}_n$.

   □

THE END

Quiz 4, STAT 316 Mathematical Statistics II, Spring 2008

Name: **Points:**

1. (10 points) Let $X_1, \ldots, X_n$ be a random sample from $N(0, \theta)$.

   (a) (5 points) Show that the distribution has monotone likelihood ratio in $\sum_{i=1}^n X_i^2$.

   (b) (5 points) Find the uniformly most powerful test with size α for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

   Solution:

   (a) Take $\theta_1 < \theta_2$ and consider the likelihood ratio $L(\theta_1; \vec{X})/L(\theta_2; \vec{X})$. After simplification, the ratio is monotonely increasing in $\sum X_i$.
   Note that this is not the same as showing the likelihood itself is monotone in $\sum X_i$.

   (b) By the Karlin-Rubin theorem, the UMP test rejects H_0 if $\sum X_i > \theta_0 \chi^2_{n;\alpha}$.

   □

2. (10 points) Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with pdf

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \; x > 0, \; \theta > 0.$$

   (a) (5 points) Find the MLE of θ.

   (b) (5 points) Derive a level α LR test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, $\theta_0 > 0$.

   Solution:

   (a) The MLE of θ is $\hat{\theta} = \sum X_i / n$.

   (b) The LR is found to be $\Lambda = \left(\frac{\sum X_i}{n\theta_0}\right)^n \exp\left(n - \frac{\sum X_i}{\theta_0}\right)$. The LR test rejects H_0 if $\Lambda < k$, which is equivalent $\sum X_i < a$ or $\sum X_i > b$, because Λ increases first and then decreases as $\sum X_i$ increases. The constants a and b determined by the size α.

   □

THE END

Quiz 5, STAT 316 Mathematical Statistics II, Spring 2008

Name: **Points:**

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1. (10 points) Suppose that $X_1, \dots, X_n$ form a random sample from $N(0, \theta)$.

   (a) (5 points) Find a variance stabilizing transformation for $\frac{\sum_{i=1}^n X_i^2}{n}$. Note that $E(X_1^4) = 3\theta^2$.

   (b) (5 points) Construct an asymptotically valid $1-\alpha$ confidence interval for θ using the variance stabilizing transformation.

   Solution:

   (a) First find $\text{Var}(X_1^2) = E(X_1^4) - E^2(X_1^2) = 2\theta^2$. By the Central Limit Theorem,

   $$\sqrt{n}\left(\frac{\sum_{i=1}^n X_i^2}{n} - \theta\right) \to N(0, 2\theta^2).$$

   The variance stabilizing transformation g needs to satisfy $g'(\theta) = \frac{1}{\sqrt{2}\theta}$. Therefore, one choice is $g(u) = \frac{1}{\sqrt{2}} \log u$.

   (b) Find the $1-\alpha$ confidence interval for $\frac{1}{\sqrt{2}} \log \theta$ first as, say, $[L, U]$. Then transform it back to the scale of θ as $[\exp(L/\sqrt{2}), \exp(U/\sqrt{2})]$.

   □

2. (10 points) Consider the Bayes model where the data $X_i|\vartheta = \theta$, $i = 1, \dots, n$, are iid Bernoulli variables with mean θ, $0 < \theta < 1$, and prior density of ϑ, $h(\theta) = 1$.

   (a) (5 points) Obtain the posterior distribution of ϑ.

   (b) (5 points) Obtain the Bayes estimate of ϑ under squared error loss.

   Solution:

   (a) This case has conjugate prior. The posterior distribution is found to be $\text{Beta}(\sum X_i + 1, n - \sum X_i + 1)$.

   (b) The Bayes estimate is the posterior mean

   $$\frac{\sum X_i + 1}{n + 2}$$

   .

   □

THE END

Solution to Midterm 1, STAT 316 — Mathematical Statistics II, Spring 2008

1. (a) Use Lehmann-Scheffe Theorem.

   (b) Note that $X_{n:n} - X_{n:1}$ is an ancillary statistic with expectation c_n. Clearly, $E[h(T_1, T_2) - c_n] = 0$, where $h(t_1, t_2) = t_1 - t_2$.

2. (a) Use variable transformation.

   (b) The distribution of T is $\Gamma(n, \alpha)$.

   (c) By differentiating the log density twice and taking negative expectation, $I_{\vec{X}}(\alpha) = n/\alpha^2$.

   (d) Since T is sufficient, $I_T(\alpha) = I_{\vec{X}}(\alpha)$.
   Alternatively, find $I_T(\alpha)$ from the fact T is a Gamma variable.

3. (a) Use variable transformation.

   (b) Solve the score equation and verify the second order condition to find $\hat{\theta} = \frac{\sum X_i^2}{2n}$.

   (c) The distribution of $\hat{\theta}$ is $\Gamma(n, \frac{\theta}{n})$.

   (d) From above, $E(\hat{\theta}) = \theta$ and $\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{n}$. $\mathrm{MSE}(\hat{\theta}) = \frac{\theta^2}{n}$.

4. (a) The independence follows from Basus's theorem by noting that $X_{n:n}$ is complete and sufficient statistic and Y_1 is ancillary.

   (b) From independence above, we have $E[X_1] = E[Y_1]E[X_{n:n}]$.

   (c) $E[X_1|X_{n:n}] = X_{n:n}E[Y_1|X_{n:n}] = X_{n:n}E[Y_1] = X_{n:n}\frac{n+1}{2n}$,

   (d) The moment estimator is $2\bar{X}$. $E[2\bar{X}|X_{n:n}] = E[2X_1|X_{n:n}] = \frac{n+1}{n}X_{n:n}$.

THE END

Solution to Midterm 2, STAT 316 — Mathematical Statistics II, Spring 2008

1. (a) The size of the test is $\alpha = \sup_{\theta \le 1} \Pr_\theta(X > 1/2) = \sup_{\theta \le 1} \int_{1/2}^{1} \theta x^{\theta-1} \mathrm{d}x = \sum_{\theta \le 1} 1 - (1/2)^\theta = 1/2$.
   (b) The power function for $\theta > 1$ is $Q(\theta) = \int_{1/2}^{1} \theta x^{\theta-1} \mathrm{d}x = 1 - (1/2)^\theta$.
   (c) First show that the distribution has MLR in X. Then by the Karlin-Rubin theorem, the UMP test rejects H_0 when $X > k$, where k satisfies $1 - k = \alpha$.
   (d) The UMP test does not exist.

2. (a) The results follows from the fact that $\frac{\bar{X}_n - \mu_0}{S/\sqrt{n}}$ is t_{n-1}.
   (b) Under H_0, the MLEs are μ_0 and $\sum_{i=1}^n (X_i - \mu_0)/n$.
   (c) The LR test, after simplification, rejects H_0 if $n(\bar{X}_n - \mu_0)^2 / \sum_{i=1}^n (X_i - \bar{X}_n)^2 > k$. See more details on page 511 of the textbook.
   (d) As $n \to \infty$, $\frac{\bar{X}_n - \mu_0}{S/\sqrt{n}}$ converges to $N(0,1)$. A size α test rejects H_0 when $|\bar{X}_n - \mu_0| > z_{\alpha/2} S/\sqrt{n}$.

3. (a) The distribution of Q is $\Gamma(n, 1)$.
   (b) Find a and b such that, for a $\Gamma(n, 1)$ variable W, $\Pr(a < W < b) = 1 - \alpha$. An $1 - \alpha$ confidence interval is obtained as $(b/\sum X_i, a/\sum X_i)$.
   (c) The expected length is $\left(\frac{1}{a} - \frac{1}{b}\right) n\lambda$.
   (d) The MLE is found to be $\hat{\theta} = \sum X_i/n$. The acceptance region of LR test turns out to be $R^c = \{\vec{x} \in R^n : k_1 < \sum X_i/\lambda_0 < k_2\}$. Inverting the acceptance gives the same confidence interval as above.

4. More details can be found in Example 7.5.4.
   (a) By Cramér-Rao inequality (Theroem 7.5.1), the Cramér-Rao lower bound is $\exp(-2\lambda)\lambda/n$.
   (b) First, $E[\hat{\theta}] = \theta$. Second, Y is complete and sufficient statistic. By Lehmann-Sheffe theorem, $E[\hat{\theta}|Y]$ is the unique UMVUE.
   (c) Note that the conditional distribution of X_1 given $Y = y$ is binomial with $n = y$ and $p = 1/n$. $E(\hat{\theta}|Y) = E[I(X_i = 0)|Y] = \left(1 - \frac{1}{n}\right)^Y$.
   (d) The MGF of Y is $\psi(t) = \exp\{n\lambda(e^t - 1)\}$. Note that $E[(1 - \frac{1}{n})^Y] = E[\exp\{\log(1 - \frac{1}{n})\}Y]$ is $\psi(\log(1 - \frac{1}{n}))$, and similarly, $E[(1 - \frac{1}{n})^{2Y}] = \psi(2\log(1 - \frac{1}{n}))$. The variance of $\hat{\theta}$ is found to be $\exp(-2\lambda)(\exp(\lambda/n) - 1)$, which does not attain the lower bound.

THE END