

Quiz 1, STAT 316 Mathematical Statistics II, Spring 2008

Name:

Points:

1. (10 points) Let X_1, \dots, X_n be an iid sample from $N(\theta, \theta)$, where $\theta > 0$ is unknown.
 - (a) (5 points) Does $N(\theta, \theta)$ belong to the exponential family? Justify.
 - (b) (5 points) Derive a minimal sufficient statistic for θ .

2. (10 points) Let X_1, \dots, X_n be an iid sample from $N(\mu, \sigma^2)$, $\mu \in R$, $\sigma^2 > 0$.
 - (a) (5 points) Find a joint sufficient statistic for (μ, σ^2) .
 - (b) (5 points) Find a sufficient statistic for σ^2 when μ is known.

THE END

Quiz 2, STAT 316 Mathematical Statistics II, Spring 2008

Name:

Points:

1. (10 points) Let X be a Bernoulli random variable with success rate $\theta \in (0, 1)$.
 - (a) (5 points) Find $I_X(\theta)$, the Fisher information about θ in X .
 - (b) (5 points) Let $\eta = g(\theta)$ be a differentiable one-to-one transformation of θ . Find $I_X(\eta)$, the Fisher information about η in X .

2. (10 points) Let X_1, \dots, X_n be an iid sample from $\Gamma(\alpha, \beta)$, where α is known. Let $T = \sum_{i=1}^n X_i$.
 - (a) (5 points) Show that $W = (X_{n:n} - X_{n:1})/T$ is an ancillary statistic.
 - (b) (5 points) Find $\text{Cov}(W, T)$.

THE END

Quiz 3, STAT 316 Mathematical Statistics II, Spring 2008

Name:

Points:

1. (10 points) Suppose that X_1, \dots, X_n form a random sample from $U(0, \theta)$. Let $X_{n:n}$ be the largest order statistic.
 - (a) (5 points) Find the UMVUE of θ^2 .
 - (b) (5 points) Find $E[X_1^2 | X_{n:n}]$.

2. (10 points) Suppose that X_1, \dots, X_n form a random sample from $N(\theta, 1)$.
 - (a) (5 points) Find the UMVUE of θ .
 - (b) (5 points) Does the variance of the UMVUE attain the Cramér-Rao lower bound?

THE END

Quiz 4, STAT 316 Mathematical Statistics II, Spring 2008

Name:

Points:

1. (10 points) Let X_1, \dots, X_n be a random sample from $N(0, \theta)$.
 - (a) (5 points) Show that the distribution has monotone likelihood ratio in $\sum_{i=1}^n X_i^2$.
 - (b) (5 points) Find the uniformly most powerful test with size α for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

2. (10 points) Let X_1, \dots, X_n be a random sample from an exponential distribution with pdf

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \quad \theta > 0.$$

- (a) (5 points) Find the MLE of θ .
- (b) (5 points) Derive a level α LR test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, $\theta_0 > 0$.

THE END

Quiz 5, STAT 316 Mathematical Statistics II, Spring 2008

Name:

Points:

1. (10 points) Suppose that X_1, \dots, X_n form a random sample from $N(0, \theta)$.
 - (a) (5 points) Find a variance stabilizing transformation for $\frac{\sum_{i=1}^n X_i^2}{n}$. Note that $E(X_1^4) = 3\theta^2$.
 - (b) (5 points) Construct an asymptotically valid $1 - \alpha$ confidence interval for θ using the variance stabilizing transformation.

2. (10 points) Consider the Bayes model where the data $X_i | \vartheta = \theta$, $i = 1, \dots, n$, are iid Bernoulli variables with mean θ , $0 < \theta < 1$, and prior density of ϑ , $h(\theta) = 1$.
 - (a) (5 points) Obtain the posterior distribution of ϑ .
 - (b) (5 points) Obtain the Bayes estimate of ϑ under squared error loss.

THE END

Midterm 1, STAT 316 — Mathematical Statistics II, Spring 2008

This is a 75-minute exam. There are 40 points in total. The problems are not necessarily in the order of difficulty level. Please watch your time. You may consult one single-sided 8.5×11 notes. Put all your work in the exam book. No credit will be given to results without necessary justification. Don't panic and good luck!

1. (10 points) Suppose that X_1, \dots, X_n are iid from $U(\theta - 1, \theta + 1)$ with unknown parameter $\theta > 0$. Let $X_{n:i}$ be the i th order statistic, $i = 1, \dots, n$. Let $T = (X_{n:1}, X_{n:n})$.

- (a) (5 points) Show that T is a minimal sufficient statistic.
(b) (5 points) Is T a complete statistic? Justify.

2. (10 points) Suppose that X_1, \dots, X_n are iid from a Weibull distribution with pdf

$$f(x; \alpha) = \frac{\beta x^{\beta-1}}{\alpha} \exp\left(-\frac{x^\beta}{\alpha}\right) I(x > 0),$$

where $\alpha > 0$ is unknown but $\beta > 0$ is known. Let $T = \sum_{i=1}^n X_i^\beta$.

- (a) (2 points) Show that $Y_1 = X_1^\beta$ is an exponential random variable with mean α .
(b) (2 points) Find the distribution of T .
(c) (3 points) Find the information $I_{\vec{X}}(\alpha)$, where \vec{X} is the whole sample.
(d) (3 points) Find the information $I_T(\alpha)$.

3. (10 points) Suppose that X_1, \dots, X_n are iid from a Rayleigh distribution with pdf

$$f(x; \theta) = \frac{x \exp(-\frac{x^2}{2\theta})}{\theta} I(x > 0),$$

where parameter $\theta > 0$.

- (a) (2 points) Show that X_1^2 has a gamma distribution with shape 1 and scale 2θ .
(b) (2 points) Find $\hat{\theta}$, the MLE of θ .
(c) (3 points) Find the distribution of $\hat{\theta}$.
(d) (3 points) Find $\text{MSE}(\hat{\theta})$.

4. (10 points) Suppose that X_1, \dots, X_n are iid from $U(0, \theta)$ with unknown parameter $\theta > 0$. Let $X_{n:n}$ be the largest order statistic.

- (a) (2 points) Argue that $Y_1 = X_1/X_{n:n}$ and $X_{n:n}$ are independent.
(b) (2 points) Show that $E[Y_1] = E[X_1]/E[X_{n:n}]$.
(c) (3 points) Find $E[X_1|X_{n:n}]$. Note that $X_1 = Y_1 X_{n:n}$.
(d) (3 points) Find the Rao-Blackwellized version of the moment estimator of θ .

THE END

Midterm 2, STAT 316 — Mathematical Statistics II, Spring 2008

This is a 75-minute exam. There are 40 points in total. The problems are not necessarily in the order of difficulty level. Please watch your time. You may consult one single-sided 8.5×11 notes. Put all your work in the exam book. No credit will be given to results without necessary justification. Don't panic and good luck!

- (10 points) Suppose that X is ONE observation from $\text{Beta}(\theta, 1)$. Consider testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$.
 - (3 points) Find the size of the test that rejects H_0 if $X > 1/2$.
 - (2 points) Express the power of the test that rejects H_0 if $X > 1/2$ using F_θ , the CDF of $\text{Beta}(\theta, 1)$.
 - (3 points) Find the UMP test of size α .
 - (2 points) Comment on the UMP test for testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$.
- (10 points) Suppose that X_1, \dots, X_n are iid from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. Let \bar{X}_n be the sample mean and S_n^2 be the sample variance. Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.
 - (2 points) Show that the test that rejects H_0 when $|\bar{X}_n - \mu_0| > t_{n-1; \alpha/2} S / \sqrt{n}$ is a test of size α .
 - (2 points) Find the MLE of μ and σ^2 under H_0 .
 - (3 points) Show that the above test can be derived as an likelihood ratio test.
 - (3 points) Assume that X_1, \dots, X_n are iid with mean μ and variance σ^2 , but not necessarily normal. As $n \rightarrow \infty$, describe an asymptotic test for H_0 with size α .
- (10 points) Suppose that X_1, \dots, X_n are iid from an exponential distribution with mean λ .
 - (2 points) Argue that $Q = \sum_{i=1}^n X_i / \lambda$ is a pivotal quantity.
 - (3 points) Construct a $1 - \alpha$ confidence interval for λ using the pivotal approach.
 - (2 points) Find the expected length of the $1 - \alpha$ confidence interval.
 - (3 points) Construct a $1 - \alpha$ confidence interval for λ by inverting the likelihood ratio test for $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$.
- (10 points) Suppose that X_1, \dots, X_n forms a random sample from a Poisson distribution with mean λ . Suppose that we are estimating $\theta = e^{-\lambda}$, the probability of $X_1 = 0$. Consider estimator $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n I(X_i = 0)$, where I is the indicator function. Let $Y = \sum_{i=1}^n X_i$.
 - (3 points) Find the Cramér-Rao lower bound for unbiased estimators of θ .
 - (3 points) Argue that $E(\hat{\theta}|Y)$ is the UMVUE of θ .
 - (2 points) Show that the UMVUE of θ is $E(\hat{\theta}|Y) = (1 - \frac{1}{n})^Y$. Note that the conditional distribution of (X_1, \dots, X_n) given $Y = y$ is a multinomial with parameters y and $(1/n, \dots, 1/n)$.
 - (2 points) Find the variance of the UMVUE and compare with the lower bound. Note that the MGF of X_1 is $\phi(t) = \exp\{\lambda(e^t - 1)\}$.

THE END

Final Exam, STAT 316 — Mathematical Statistics II, Spring 2008

This is a 120-minute exam. There are 60 points in total. The problems are not necessarily in the order of difficulty level. Please watch your time. You may consult six single-sided 8.5×11 notes. Put all your work in the exam book. No credit will be given to results without necessary justification. Don't panic and good luck!

- (15 points) Suppose that $X_i|\vartheta = \theta$ are iid $\Gamma(1, \theta)$, $i = 1, \dots, n$, where $\theta > 0$ is the rate (not scale) parameter. Consider an improper prior distribution of ϑ such that the density $h(\theta) \propto \frac{1}{\theta}$.
 - (3 points) Show that $h(\theta) \propto \sqrt{I_X(\theta)}$, where I_X is the information of X_1 .
 - (3 points) Find the posterior distribution of ϑ .
 - (3 points) Find the Bayes estimator of θ under the squared error loss.
 - (3 points) Construct a $1 - \alpha$ credible interval for θ .
 - (3 points) Describe how to perform a Bayes test for $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$, where θ_0 is specified.
- (15 points) Suppose that X_1, \dots, X_n, X_{n+1} are iid $N(\mu, \sigma^2)$, with σ^2 known, $n \geq 2$. Consider the parametric function $\tau(\mu) = \Pr(\sum_{i=1}^n X_i > X_{n+1})$.
 - (3 points) Argue that $U = \sum_{i=1}^{n+1} X_i$ is complete and sufficient for μ .
 - (3 points) Let $W = \sum_{i=1}^n X_i - X_{n+1}$. Find the joint distribution of (W, U) .
 - (3 points) Let $T = I(W > 0)$. Argue that $E[T|U]$ is the UMVUE of $\tau(\mu)$.
 - (3 points) Find $E[T|U]$.
 - (3 points) Find $E[X_1^2|U]$ using the Lehmann-Scheffé theorems.

- (15 points) Suppose that X_1, \dots, X_n are iid from a Weibull distribution with pdf

$$f(x; \alpha) = \frac{\beta x^{\beta-1}}{\alpha} \exp\left(-\frac{x^\beta}{\alpha}\right) I(x > 0),$$

where $\alpha > 0$ is unknown but $\beta > 0$ is known. Let $T = \sum_{i=1}^n X_i^\beta$.

- (3 points) Show that $Y_1 = X_1^\beta$ is an exponential random variable with mean α .
 - (3 points) Show that the MLE of α is $\hat{\alpha} = T/n$.
 - (3 points) Identify the distribution of $\hat{\alpha}$.
 - (3 points) Find the information $I_{\vec{X}}(\alpha)$, where \vec{X} is the whole sample.
 - (3 points) Does the variance of the MLE attain the Cramér-Rao lower bound?
- (15 points) Suppose that X_1, \dots, X_n are iid from a pdf $f(x; \mu_1, \sigma)$ and Y_1, \dots, Y_n are iid from a pdf $f(x; \mu_2, \sigma)$, where $f(x; \mu, \sigma) = \sigma^{-1} \exp\{-(x - \mu)/\sigma\}$, $x > \mu$. Assume that σ is known. Let $X_{n:1}$ and $Y_{n:1}$ be the first order statistics from the two samples, respectively. It is known that if Z_1 and Z_2 are iid standard exponential variables, then $Z_1 - Z_2$ is a standard Laplace (or double exponential) variable.
 - (3 points) Show that $n(X_{n:1} - \mu_1)/\sigma$ is a standard exponential variable.
 - (3 points) Argue that $X_{n:1} - Y_{n:1} - (\mu_1 - \mu_2)$ is a pivotal quantity.
 - (3 points) Construct a $1 - \alpha$ confidence interval for $\mu_1 - \mu_2$.
 - (3 points) Design a test for $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$ with significance level α .
 - (3 points) Comment on the acceptance region of the test and the confidence interval.

THE END