Quiz 5 Solution, STAT 5685 — Mathematical Statistics II, Spring 2009

Name: **Points:**

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1. (10 points) Suppose that $X_1, \ldots, X_n$ are iid from a Poisson distribution with mean θ. Suppose that the prior density of θ is $h(\theta) \propto \theta^{-1/2}$.

   (a) (5 points) Find the posterior distribution of θ.

   (b) (5 points) Construct a $1-\alpha$ credible interval for θ.

   Solution:

   (a) The posterior density is

   $$k(\theta|x_1, \ldots, x_n) \propto \theta^{-1/2} \exp(-n\theta)\theta^{\sum_{i=1}^n x_i},$$

   which is recognized as $\Gamma(t + \frac{1}{2}, n)$, where $t = \sum_{i=1}^n x_i$.

   (b) A $1-\alpha$ credible interval is (a, b), where a and b are the $\alpha/2$ and $1-\alpha/2$ quantiles of $\Gamma(t+\frac{1}{2}, n)$, where n is the rate parameter.

   A HPD credible interval is (c, d), where c and d are such that the probability of a $\Gamma(t + \frac{1}{2}, n)$ variable falls between them is $1-\alpha$, and the density at them are equal.

   □

2. (10 points) Suppose that $X_1, \ldots, X_n$ are iid from $N(\mu, 1/\tau)$ where μ is known and the precision τ is unknown. Suppose that the prior distribution of τ is $\Gamma(\alpha, \beta)$ where β is the rate parameters.

   (a) (5 points) Find the posterior distribution of τ.

   (b) (5 points) Find the Bayes estimator of τ under squared error loss.

   Solution:

   (a) The posterior density is

   $$k(\tau|x_1, \ldots, x_n) \propto \tau^{n/2} \exp\left(-\frac{\tau}{2}\sum_{i=1}^n (x_i-\mu)^2\right) \tau^{\alpha-1}\exp(-\beta\tau) = \tau^{n/2+\alpha-1}\exp\left(-(\beta + \frac{1}{2}(x_i-\mu)^2)\tau\right),$$

   which is recognized as $\Gamma(n/2+\alpha, \beta + \frac{1}{2}\sum_{i=1}^n (x_i-\mu)^2)$.

   (b) The Bayes estimator is posterior mean,

   $$\frac{n/2+\alpha}{\beta + \frac{1}{2}\sum_{i=1}^n (x_i-\mu)^2}.$$

   □

THE END